

# Strong coupling asymptotics for a singular Schrödinger operator with an interaction supported by an open arc

PAVEL EXNER <sup>†</sup> and KONSTANTIN PANKRASHKIN <sup>‡</sup>

<sup>†</sup> Department of Theoretical Physics, Nuclear Physics Institute  
Czech Academy of Sciences, 25068 Řež near Prague, Czechia  
Doppler Institute for Mathematical Physics  
and Applied Mathematics  
Czech Technical University, Břehová 7, 11519 Prague, Czechia  
E-mail: [exner@ujf.cas.cz](mailto:exner@ujf.cas.cz)

<sup>‡</sup> Laboratoire de mathématiques d'Orsay, UMR 8628  
Université Paris-Sud 11, Bâtiment 425, 91400 Orsay, France  
E-mail: [konstantin.pankrashkin@math.u-psud.fr](mailto:konstantin.pankrashkin@math.u-psud.fr)

## Abstract

We consider a singular Schrödinger operator in  $L^2(\mathbb{R}^2)$  written formally as  $-\Delta - \beta\delta(x - \gamma)$  where  $\gamma$  is a  $C^4$  smooth open arc in  $\mathbb{R}^2$  of length  $L$  with regular ends. It is shown that the  $j$ th negative eigenvalue of this operator behaves in the strong-coupling limit,  $\beta \rightarrow +\infty$ , asymptotically as

$$E_j(\beta) = -\frac{\beta^2}{4} + \mu_j + \mathcal{O}\left(\frac{\log \beta}{\beta}\right),$$

where  $\mu_j$  is the  $j$ th Dirichlet eigenvalue of the operator

$$-\frac{d^2}{ds^2} - \frac{\kappa(s)^2}{4}$$

on  $L^2(0, L)$  with  $\kappa(s)$  being the signed curvature of  $\gamma$  at the point  $s \in (0, L)$ .

## 1 Introduction

Singular Schrödinger operators with interaction supported by manifolds of a lower dimension have been a subject of investigation in numerous papers,

particularly in the last decade. One motivation came from physics where operators formally written as

$$-\Delta - \beta\delta(x - \gamma)$$

with  $\beta > 0$ , where  $\gamma$  is metric graph embedded in a Euclidean space, are used as models of ‘leaky quantum graphs’ describing motion of particles confined to a graph in a way allowing quantum tunneling between different parts of  $\gamma$ . At the same time there is a mathematical motivation to study such operators because they exhibit nontrivial and interesting relations between spectral properties and the geometry of the interaction support. In the informal language, the above operator is the Laplacian with the boundary conditions on  $\gamma$ ,  $[\partial f] + \beta f = 0$ , where  $[\partial f]$  denotes the jump of the normal derivative of  $f$  on  $\gamma$ ; the rigorous definition is given by the associated sesquilinear form [3], see below, and the boundary conditions should be understood in a certain weak sense.

An overview of known results concerning leaky quantum graphs is given in [5] which also offers a number of open problems. Some of them concern the *strong-coupling behavior* of such operators. For large  $\beta$  one expects the eigenfunctions corresponding to eigenvalues at the bottom of the spectrum to be strongly concentrated around  $\gamma$  which suggests the asymptotic spectral behaviour might be determined by a one-dimensional problem.

In the simplest case when we consider the indicated operator in  $L^2(\mathbb{R}^2)$  and  $\gamma$  is a sufficiently smooth curve without self-intersections and endpoints — either an infinite one with a suitable asymptotic behaviour or a loop — such result is indeed known [5, 6]: the eigenvalues at the bottom of the spectrum diverge as  $-\frac{1}{4}\beta^2$  but the next term in the expansion is the respective eigenvalue of a one-dimensional Schrödinger operator with a potential determined by the curvature of  $\gamma$ . We note that the smoothness hypothesis is essential; the asymptotics is expected to be completely different, e.g., if  $\gamma$  has corners, cf. [8].

One asks naturally how such an asymptotics could look like if the curve has endpoints and one has to impose boundary conditions to make the corresponding one-dimensional Schrödinger operator self-adjoint. Note that the Hamiltonian in question can be viewed as a special type mixed problem, cf. e.g. [2, 7].

A conjecture was made in Sec. 7.12 of [5] that under proper regularity assumptions it is the Dirichlet condition which gives the asymptotics. The aim of the present paper is to prove this conjecture in the case when  $\gamma$  is a  $C^4$  smooth arc in  $\mathbb{R}^2$  with regular endpoints. A precise formulation of this result is stated in the next section and the rest of the paper is devoted to the proof.

As in the case of a curve without endpoints we employ a bracketing argument imposing Dirichlet and Neumann condition at the boundary of a tubular

neighbourhood of  $\gamma$ . In the present case, however, we need a neighbourhood extending beyond the endpoints and we lose the asymptotic separation of variables employed in [6]. Instead we have to establish the decay of eigenfunctions away of  $\gamma$  which is technically the main part of the proof.

## 2 Main result

Let  $\gamma$  be an open  $C^4$  arc in  $\mathbb{R}^2$  of length  $L > 0$  and with regular ends. More precisely, we assume that, for some  $l_0 > 0$ , there is an injective  $C^4$  function  $\Gamma : [-l_0, L + l_0] \ni s \mapsto (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2$  satisfying at any point  $|\Gamma'(s)| = 1$ , and the arc  $\gamma$  is identified with  $\Gamma((0, L))$ . Denote by  $\kappa(s)$  the signed curvature of  $\gamma$  at  $\Gamma(s)$ , i.e.

$$\kappa(s) := \Gamma_1'(s)\Gamma_2''(s) - \Gamma_1''(s)\Gamma_2'(s).$$

Let  $\beta > 0$ . Consider the sesquilinear form  $h_\beta$  defined on  $H^1(\mathbb{R}^2)$  by

$$h_\beta(f, f) = \iint_{\mathbb{R}^2} |\nabla f|^2 dx - \beta \int_{\gamma} |f(x)|^2 dS,$$

and let  $H_\beta$  be the self-adjoint operator in  $L^2(\mathbb{R}^2)$  associated with  $h_\beta$ . Since  $\gamma$  has a finite length, it is easy to see that the essential spectrum of  $H_\beta$  is  $[0, +\infty)$ . Denote by  $E_1(\beta) \leq E_2(\beta) \leq \dots E_j(\beta) \leq \dots$  the negative eigenvalues of  $H_\beta$  with their multiplicities taken into account. Our main result reads as follows:

**Theorem 1.** *For any  $j \in \mathbb{N}$ , the asymptotic expansion*

$$E_j(\beta) = -\frac{\beta^2}{4} + \mu_j + \mathcal{O}\left(\frac{\log \beta}{\beta}\right),$$

*holds for strong coupling,  $\beta \rightarrow +\infty$ , where  $\mu_j$  is the  $j$ th Dirichlet eigenvalue of the Schrödinger operator*

$$-\frac{d^2}{ds^2} - \frac{\kappa(s)^2}{4}$$

*with curvature-induced potential on  $[0, L]$ .*

## 3 Scheme of the proof

We put

$$\tau(s) := \begin{pmatrix} \Gamma_1'(s) \\ \Gamma_2'(s) \end{pmatrix}, \quad n(s) := \begin{pmatrix} -\Gamma_2'(s) \\ \Gamma_1'(s) \end{pmatrix};$$

in other words  $\tau(s)$  is a unit tangent vector and  $n(s)$  is a unit normal vector to  $\gamma$  at the point  $\Gamma(s)$ , by assumption both continuously depending on the arc-length parameter, not only on the arc itself but also on the extensions beyond its endpoints, i.e. for  $s \in [-l_0, L + l_0]$ . In what follows we denote

$$K := \|\kappa\|_{L^\infty(-l_0, L+l_0)}.$$

For any  $\alpha \in (0, l_0)$  let us introduce the following subdomains in  $\mathbb{R}^2$ :

$$P(\alpha) := (-\alpha, L + \alpha) \times (-\alpha, \alpha), \quad (1)$$

$$\Omega(\alpha) := \{\Gamma(s) + tn(s) : (s, t) \in (0, L) \times (-\alpha, \alpha)\}, \quad (2)$$

$$\Pi(\alpha) := \{\Gamma(s) + tn(s) : (s, t) \in P(\alpha)\} \quad (3)$$

and the prolonged arc

$$\gamma_a := \Gamma((-a, L + a)) \subset \Pi(a).$$

Clearly,  $\gamma \subset \gamma_a$  for any  $a > 0$ . Furthermore, one can check in a similar way as in [6] that there is  $a_0 \in (0, \frac{1}{2K})$  such that the map

$$P(a) \ni (s, t) \mapsto \Phi(s, t) = \Gamma(s) + tn(s) \in \Pi(a) \quad (4)$$

is a diffeomorphism for any fixed  $a \in (0, a_0]$ . Throughout the rest of the paper we will always use

$$a = \frac{6 \log \beta}{\beta}. \quad (5)$$

Let us introduce the following sesquilinear forms:

$$h_{\beta,a} = \iint_{\Pi(a)} |\nabla f|^2 dx - \beta \int_{\gamma} |f|^2 dS, \quad f \in H_0^1(\Pi(a)), \quad (6)$$

$$\tilde{h}_{\beta,a} = \iint_{\Omega(a)} |\nabla f|^2 dx - \beta \int_{\gamma} |f|^2 dS, \quad f \in H_0^1(\Omega(a)), \quad (7)$$

$$\hat{h}_{\beta,a} = \iint_{\Pi(a)} |\nabla f|^2 dx - \beta \int_{\gamma_a} |f|^2 dS, \quad f \in H_0^1(\Pi(a)), \quad (8)$$

and denote the associated self-adjoint operators, acting respectively in  $L^2(\Pi(a))$ ,  $L^2(\Omega(a))$  and  $L^2(\Pi(a))$ , by  $L_\beta$ ,  $\tilde{L}_\beta$ ,  $\hat{L}_\beta$ . We consider their eigenvalues  $\Lambda_j(\beta)$ ,  $\tilde{\Lambda}_j(\beta)$ ,  $\hat{\Lambda}_j(\beta)$  enumerated in the non-decreasing order taking their multiplicities into account; by the max-min principle we have

$$E_j(\beta) \leq \Lambda_j(\beta). \quad (9)$$

The asymptotic behavior of the right-hand side can be found easily:

**Proposition 2.** *For all sufficiently large  $\beta$  one has*

$$\Lambda_j(\beta) = -\frac{\beta^2}{4} + \mu_j + O\left(\frac{\log \beta}{\beta}\right).$$

**Proof.** Due to the max-min principle for any  $j \in \mathbb{N}$  we have

$$\widehat{\Lambda}_j(\beta) \leq \Lambda_j(\beta) \leq \widetilde{\Lambda}_j(\beta).$$

Furthermore, the asymptotics of the estimating eigenvalues  $\widetilde{\Lambda}_j$  and  $\widehat{\Lambda}_j$  can be obtained using the technique introduced in [6], that is, an asymptotic separation of variables:

$$\widetilde{\Lambda}_j(\beta) = -\frac{\beta^2}{4} + \mu_j + O\left(\frac{\log \beta}{\beta}\right), \quad \widehat{\Lambda}_j(\beta) = -\frac{\beta^2}{4} + \mu_j(\beta) + O\left(\frac{\log \beta}{\beta}\right),$$

where  $\mu_j$  and  $\mu_j(\beta)$  are the  $j$ th Dirichlet eigenvalues of the operators acting as

$$-\frac{d^2}{ds^2} - \frac{\kappa(s)^2}{4}$$

on  $[0, L]$  and  $[-a, L + a]$ , respectively; recall that  $a$  depends on  $\beta$ . As the Dirichlet eigenvalues are  $C^1$  functions of the interval edges, see e.g. [4], we have  $\mu_j(\beta) = \mu_j + \mathcal{O}(a) = \mu_j + O\left(\frac{\log \beta}{\beta}\right)$ , which proves the result.  $\square$

Hence the claim of Theorem 1 will be a consequence of the following asymptotic relation:

**Proposition 3.** *For any  $j \in \mathbb{N}$  one has*

$$\Lambda_j(\beta) - E_j(\beta) = \mathcal{O}\left(\frac{\log \beta}{\beta}\right)$$

*as the coupling parameter  $\beta$  tends to  $+\infty$ .*

This is our main estimate and the rest of the paper will be dedicated to the proof of Proposition 3.

## 4 Technical estimates

We denote by  $d(x, \gamma)$  the distance between a point  $x \in \mathbb{R}^2$  and the arc  $\gamma$ . In the present section we give some expressions of  $d(x, \gamma)$  for  $x \in \Pi(a)$  which we need in the following. Some of the formulæ are known, but we prefer to collect all the necessary information in this section for the sake of completeness.

Recall first the Frenet formulæ

$$\tau'(s) = \kappa(s)n(s), \quad n'(s) = -\kappa(s)\tau(s). \quad (10)$$

In particular, for all  $(s, t), (s', t') \in P(a_0)$  one has the representations

$$\Gamma(s') = \Gamma(s) + (s' - s)\tau(s) + (s' - s)^2\rho_1(s', s), \quad (11)$$

$$n(s') = n(s) - (s' - s)\kappa(s)\tau(s) + (s' - s)^2\rho_2(s', s), \quad (12)$$

$$\tau(s') = \tau(s) + (s' - s)\kappa(s)n(s) + (s' - s)^2\rho_3(s', s), \quad (13)$$

$$\begin{aligned} \Phi(s', t') &= \Phi(s, t) + (s' - s)(1 - t'\kappa(s))\tau(s) + (t' - t)n(s) \\ &\quad + (s' - s)^2(\rho_1(s', s) + t'\rho_2(s', s)) \\ &= \Phi(s, t) + (s' - s)(1 - t'\kappa(s))\tau(s) + (t' - t)n(s) \\ &\quad + (s' - s)^2\rho_4(s, t, s', t'), \end{aligned} \quad (14)$$

with

$$\rho_1, \rho_2, \rho_3 \in L^\infty((-a_0, L + a_0)^2), \quad \rho_4 \in L^\infty(P(a_0)^2).$$

**Lemma 4.** *Let  $\alpha \in (0, a_0)$ . Then there are  $C_1, C_2 > 0$  such that*

$$C_1((s - s')^2 + (t - t')^2) \leq |\Phi(s, t) - \Phi(s', t')|^2 \leq C_2((s - s')^2 + (t - t')^2) \quad (15)$$

*holds for all  $(s, t), (s', t') \in P(\alpha)$ .*

**Proof.** We have  $\overline{P(\alpha)} \subset P(a_0)$  and  $\overline{\Pi(\alpha)} \subset \Pi(a_0)$ . The upper bound in (15) follows then from the boundedness of the partial derivatives of  $\Phi$  on  $\overline{P(\alpha)}$ . Let us prove the lower one.

Suppose that the inequality is not valid, then one can find sequences  $(s_n, t_n), (s'_n, t'_n) \subset P(\alpha)$  such that, for all  $n \in \mathbb{N}$ ,

$$|\Phi(s'_n, t'_n) - \Phi(s_n, t_n)|^2 < \frac{r_n}{n}, \quad r_n := (s'_n - s_n)^2 + (t'_n - t_n)^2. \quad (16)$$

As  $\overline{P(\alpha)}$  is compact, without loss of generality we may assume that both the sequences converge,  $(s_n, t_n) \rightarrow (s, t)$  and  $(s'_n, t'_n) \rightarrow (s', t')$  as  $n \rightarrow \infty$  with some  $(s, t), (s', t') \in \overline{P(\alpha)}$ , and by (16) one has  $\Phi(s, t) = \Phi(s', t')$ . As  $\Phi$  is a diffeomorphism between  $P(a_0) \supset \overline{P(\alpha)}$  and  $\Pi(a_0)$ , one has  $(s, t) = (s', t')$ , and consequently  $\lim_{n \rightarrow \infty} r_n = 0$ . On the other hand, using the representation (14) and the fact that  $\tau(s)$  and  $n(s)$  are unit vectors, we get

$$|\Phi(s'_n, t'_n) - \Phi(s_n, t_n)|^2 = (1 - t'_n\kappa(s_n))^2(s'_n - s_n)^2 + (t'_n - t_n)^2 + \mathcal{O}(r_n^{3/2}).$$

We have  $|t'_n| < \alpha < a_0 < \frac{1}{2K}$  for any  $n$ , and choosing  $n$  large enough (hence having  $r_n$  small), we obtain

$$|\Phi(s'_n, t'_n) - \Phi(s_n, t_n)|^2 \geq \frac{r_n}{4}$$

which contradicts, however, to relation (16).  $\square$

**Lemma 5.** *There exists  $\alpha \in (0, a_0)$  such that  $d(\Phi(s, t), \gamma) = |t|$  holds for all  $(s, t) \in (0, L) \times (-\alpha, \alpha)$ .*

**Proof.** Let us pick  $(s, t) \in (0, L) \times (-\alpha, \alpha)$  and consider the function  $f : (0, L) \rightarrow \mathbb{R}_+$  defined by

$$f(\sigma) = |\Phi(s, t) - \Phi(\sigma, 0)|^2 = |\Gamma(\sigma) - \Gamma(s)|^2 - 2tn(s) \cdot (\Gamma(\sigma) - \Gamma(s)) + t^2.$$

Using again the fact that  $|\tau(\sigma)| = 1$  we find

$$\begin{aligned} f'(\sigma) &= 2\tau(\sigma) \cdot (\Gamma(\sigma) - \Gamma(s)) - 2tn(s) \cdot \tau(\sigma), \\ f''(\sigma) &= 2k(\sigma)n(\sigma) \cdot (\Gamma(\sigma) - \Gamma(s)) + 2 - 2tk(\sigma)n(s) \cdot n(\sigma), \end{aligned}$$

in particular,  $f(s) = t^2$ ,  $f'(s) = 0$  and  $f''(s) = 2 - 2t\kappa(s)$ . Hence one can choose  $\alpha_1$  sufficiently small to have  $f''(s) > 1$  for all  $(s, t) \in (0, L) \times (-\alpha_1, \alpha_1)$ , in which case  $s$  is a local minimum of  $f$ . Note also that  $f'''$  is bounded. Therefore, using the Taylor expansion, we see that one can find  $\delta_1 > 0$  such that

$$|\Phi(s, t) - \Phi(\sigma, 0)|^2 \geq t^2 + \frac{(s - \sigma)^2}{4} > t^2 \quad (17)$$

for all  $(s, t) \in (0, L) \times (-\alpha_1, \alpha_1)$  and all  $\sigma$  with  $0 < |s - \sigma| < \delta_1$ .

On the other hand, we infer from Lemma 4 that there are  $\alpha_2 > 0$  and  $c > 0$  such that

$$|\Phi(s, t) - \Phi(\sigma, 0)|^2 \geq c^2(t^2 + (s - \sigma)^2) \geq c^2(s - \sigma)^2 \quad (18)$$

holds for all  $(s, t) \in (0, L) \times (-\alpha_2, \alpha_2)$  and all  $\sigma \in (0, L)$ .

Choosing now  $\alpha < \min(c\delta_1, \alpha_1)$ , we get for any  $(s, t) \in (0, L) \times (-\alpha, \alpha)$  the following alternatives

$$|\Phi(s, t) - \Phi(\sigma, 0)| \begin{cases} \geq c\delta_1 > \alpha > |t|, & |s - \sigma| \geq \delta_1 & \text{by (18),} \\ > |t|, & 0 < |s - \sigma| < \delta_1 & \text{by (17),} \\ = |t|, & \sigma = s, \end{cases}$$

which concludes the proof.  $\square$

**Lemma 6.** *There exists  $\alpha \in (0, a_0)$  such that*

$$d(\Phi(s, t), \gamma) = \begin{cases} |\Phi(s, t) - \Phi(0, 0)|, & (s, t) \in (-\alpha, 0) \times (-\alpha, \alpha), \\ |\Phi(s, t) - \Phi(L, 0)|, & (s, t) \in (L, L + \alpha) \times (-\alpha, \alpha). \end{cases}$$

**Proof.** We will prove the first equality only, the second one can be demonstrated in a similar way. Pick  $(s, t) \in (-a_0, 0) \times (-a_0, a_0)$  and consider the function  $f_{s,t} : (0, L) \rightarrow \mathbb{R}_+$ ,

$$f_{s,t}(\sigma) = |\Phi(s, t) - \Phi(\sigma, 0)|^2 \equiv |\Gamma(s) - \Gamma(\sigma) + tn(s)|^2.$$

Using (11), (12), (13) and denoting  $\kappa_0 := \kappa(0)$ ,  $\tau_0 = \tau(0)$ ,  $n_0 := n(0)$  we have

$$\begin{aligned}\Gamma(s) &= \Gamma(0) + s\tau_0 + s^2\rho_1(s, 0), \\ \Gamma(\sigma) &= \Gamma(0) + \sigma\tau_0 + \sigma^2\rho_1(\sigma, 0), \\ n(s) &= n_0 - s\kappa_0\tau_0 + s^2\rho_2(s, 0), \\ \tau(\sigma) &= \tau_0 + \sigma\kappa_0n_0 + \sigma^2\rho_3(\sigma, 0),\end{aligned}$$

which gives

$$\begin{aligned}f'_{s,t}(\sigma) &= -2\tau(\sigma) \cdot (\Gamma(s) - \Gamma(\sigma) + tn(s)) \\ &= -2(\tau_0 + \sigma\kappa_0n_0 + \sigma^2\rho_3(\sigma)) \\ &\quad \cdot ((1 - t\kappa_0)s - \sigma)\tau_0 + tn_0 + s^2\rho_1(s, 0) - \sigma^2\rho_1(\sigma, 0) + ts^2\rho_2(s, 0).\end{aligned}$$

Using the orthogonality of  $\tau_0$  and  $n_0$ , this can be rewritten in the form

$$f'_{s,t}(\sigma) = 2(1 - t\kappa_0)(\sigma - s) + s^2A(s, t, \sigma) + \sigma^2B(s, t, \sigma),$$

where  $A$  and  $B$  are certain bounded functions. Hence one can choose  $a_1 \in (0, a_0)$  such that for all  $(s, t) \in (-a_1, 0) \times (-a_1, a_1)$  and all  $\sigma \in (0, a_1)$  one has  $f'_{s,t}(\sigma) > 0$ , and consequently

$$f_{s,t}(0) = \inf_{\sigma \in (0, a_1)} f_{s,t}(\sigma).$$

Next one can find  $a_2 \in (0, a_1)$  such that

$$B(\Phi(0, 0), a_2) \cap \gamma = B(\Phi(0, 0), a_2) \cap \Gamma((0, a_1)),$$

and finally we take  $\alpha \in (0, a_2)$  such that

$$\Phi((-\alpha, 0) \times (-\alpha, \alpha)) \subset B\left(\Phi(0, 0), \frac{a_2}{2}\right).$$

For any  $(s, t) \in (-\alpha, 0) \times (-\alpha, \alpha)$  we infer now, using the monotonicity of the associated function  $f_{s,t}$ , that

$$\begin{aligned}d(\Phi(s, t), \gamma) &= \inf_{x \in \gamma} |\Phi(s, t) - x| \\ &= \inf_{x \in B(\Phi(0, 0), \alpha) \cap \Gamma((0, a_1))} |\Phi(s, t) - x| \\ &= \inf_{\substack{\sigma \in (0, a_1): \\ \Gamma(\sigma) \in B(\Phi(0, 0), \alpha)}} |\Phi(s, t) - \Phi(\sigma, 0)| \\ &= \inf_{\sigma \in (0, a_1)} |\Phi(s, t) - \Phi(\sigma, 0)| = |\Phi(s, t) - \Phi(0, 0)|.\end{aligned}$$

□



**Lemma 7.** For  $s < 0$  we have in the limit  $(s, t) \rightarrow 0$  the relation

$$d(\Phi(s, t), \gamma) = \sqrt{s^2 + t^2} + \mathcal{O}(s^2 + t^2). \quad (19)$$

Similarly, for  $s > L$  and  $(s, t) \rightarrow (L, 0)$  we have

$$d(\Phi(s, t), \gamma) = \sqrt{(s - L)^2 + t^2} + \mathcal{O}((s - L)^2 + t^2).$$

**Proof.** We again limit ourselves to checking the first relation; the proof of the second one is analogous. By Lemma 6, for  $(s, t)$  close to  $(0, 0)$  with  $s < 0$  one has

$$\begin{aligned} d(\Phi(s, t), \gamma)^2 &= |\Gamma(s) - \Gamma(0) + tn(s)|^2 \\ &= |s\tau_0 + s^2\rho_1(s, 0) + tn_0 - ts\kappa_0\tau_0 + ts^2\rho_2(s, 0)|^2 \\ &= s^2 + t^2 + ts^2A(s, t) + s^3B(s, t) \end{aligned}$$

with some bounded functions  $A$  and  $B$ , where we have again employed the orthogonality of  $\tau_0$  and  $n_0$ . Hence we have

$$d(\Phi(s, t), \gamma)^2 = (s^2 + t^2) \left( 1 + \mathcal{O}(\sqrt{s^2 + t^2}) \right),$$

which yields the relation (19). □

Applying Lemmata 5 and 7 to the boundary of  $\Pi(a)$  we obtain

**Corollary 8.** There are  $\alpha_0 \in (0, a_0)$  and  $C > 0$  such that

$$d(x, \gamma) \geq \alpha - C\alpha^2$$

holds for all  $\alpha \in (0, \alpha_0)$  and  $x \in \partial\Pi(\alpha)$ .

For a fixed  $b > 0$  we introduce the set

$$W(b) = \{x : d(x, \gamma) < b\}.$$

and derive an integral estimate on the complement of such a neighborhood:

**Lemma 9.** Let  $k, c > 0$ . In the limit  $\beta \rightarrow +\infty$  we have

$$\iint_{\mathbb{R}^2 \setminus W\left(\frac{k \log \beta - c}{\beta}\right)} e^{-(\beta - \log \beta)d(x, \gamma)} dx = \mathcal{O}\left(\frac{1}{\beta^{k+1}}\right).$$

**Proof.** During the demonstration we denote by  $C_j$  various fixed positive numbers. Pick  $p \in (0, 1)$  with  $p > \sqrt{\frac{k-1}{k}}$ . Then by Lemmata 5 and 7 one can find  $\alpha > 0$  such that

- $d(\Phi(s, t), \gamma) = |t|$  holds for all  $s \in (0, L)$  and  $t \in (-\alpha, \alpha)$ ,
- $p\sqrt{s^2 + t^2} \leq d(\Phi(s, t), \gamma) \leq p^{-1}\sqrt{s^2 + t^2}$  holds for all  $s \in (-\alpha, 0)$  and  $t \in (-\alpha, \alpha)$ , and similarly,
- $p\sqrt{(s - L)^2 + t^2} \leq d(\Phi(s, t), \gamma) \leq p^{-1}\sqrt{(s - L)^2 + t^2}$  holds for all  $s \in (L, L + \alpha)$  and  $t \in (-\alpha, \alpha)$ .

One can represent the integration domain as follows:

$$\begin{aligned} \mathbb{R}^2 \setminus W\left(\frac{k \log \beta - c}{\beta}\right) &= \left[ W(\alpha) \setminus W\left(\frac{k \log \beta - c}{\beta}\right) \right] \\ &\cup \left[ W(2L) \setminus W(\alpha) \right] \\ &\cup \left[ \mathbb{R}^2 \setminus W(2L) \right]. \end{aligned}$$

Let us estimate the contribution to the integral from each of these three components. Using the diffeomorphism  $\Phi$  one easily reduces the integration on  $W(\alpha) \setminus W\left(\frac{k \log \beta - c}{\beta}\right)$  to the integration on two rectangles and two half-discs: this yields the estimate

$$\begin{aligned} &\iint_{W(\alpha) \setminus W\left(\frac{k \log \beta - c}{\beta}\right)} e^{-(\beta - \log \beta)d(x, \gamma)} dx \\ &\leq C_1 \iint_{p \frac{k \log \beta - c}{\beta} \leq |x| \leq p^{-1}\alpha} e^{-p(\beta - \log \beta)|x|} dx + C_2 \int_0^L \int_{\frac{k \log \beta - c}{\beta}}^\alpha e^{-(\beta - \log \beta)t} dt ds \\ &\leq C_3 \int_{p \frac{k \log \beta - c}{\beta}}^{p^{-1}\alpha} r e^{-p(\beta - \log \beta)r} dr + C_4 \int_{\frac{k \log \beta - c}{\beta}}^\alpha e^{-(\beta - \log \beta)t} dt. \end{aligned}$$

We have

$$\int_{\frac{k \log \beta - c}{\beta}}^\alpha e^{-(\beta - \log \beta)t} dt = \frac{\beta^{-k} e^{c + \frac{k \log^2 \beta}{\beta} - c \frac{\log \beta}{\beta}} - e^{-\alpha(\beta - \log \beta)}}{\beta - \log \beta} = \mathcal{O}\left(\frac{1}{\beta^{k+1}}\right)$$

and similarly,

$$\int_{p \frac{k \log \beta - c}{\beta}}^{p^{-1}\alpha} r e^{-p(\beta - \log \beta)r} dr = \mathcal{O}\left(\frac{\log^2 \beta}{\beta^{p^2 k + 2}}\right) = \mathcal{O}\left(\frac{1}{\beta^{k+1}}\right).$$

Putting these estimates together we find

$$\iint_{W(\alpha) \setminus W\left(\frac{k \log \beta - c}{\beta}\right)} e^{-(\beta - \log \beta)d(x, \gamma)} dx = \mathcal{O}\left(\frac{1}{\beta^{k+1}}\right).$$

Furthermore, the measure of the second component,  $W(2L) \setminus W(\alpha)$ , certainly does not exceed  $9\pi L^2$ , while for all  $x$  in this domain the integrated function is majorized by  $e^{-(\beta - \log \beta)d(x, \gamma)} \leq \beta^\alpha e^{-\alpha\beta}$ , which gives

$$\iint_{W(2L) \setminus W(\alpha)} e^{-(\beta - \log \beta)d(x, \gamma)} dx \leq 9\pi L^2 \beta^\alpha e^{-\alpha\beta} = \mathcal{O}(e^{-\alpha\beta/2}).$$

Finally, to estimate the integral over the complement of  $W(2L)$  let us pick a point  $x_0 \in \gamma$  and consider  $x \notin W(2L)$ . One has

$$d(x, \gamma) \geq |x - x_0| - L \geq |x - x_0| - \frac{|x - x_0|}{2} = \frac{|x - x_0|}{2}.$$

Hence we have

$$\begin{aligned} \iint_{\mathbb{R}^2 \setminus W(2L)} e^{-(\beta - \ln \beta)d(x, \gamma)} dx &\leq \iint_{|x - x_0| > 2L} e^{-(\beta - \ln \beta)|x - x_0|/2} dx \\ &= 2\pi \int_{2L}^{\infty} r e^{-(\beta - \ln \beta)r/2} dr = 2\pi \left( \frac{4L}{\beta - \log \beta} + \frac{4}{(\beta - \log \beta)^2} \right) \beta^L e^{-L\beta} \\ &= \mathcal{O}(e^{-L\beta/2}), \end{aligned}$$

and summing up the three terms one obtains the sought result.  $\square$

## 5 Eigenfunctions estimates

Let us give first a rough *a priori* estimate for the eigenvalues  $E_j(\beta)$ .

**Lemma 10.** *For any  $j \in \mathbb{N}$  one has*

$$\frac{\beta - \ln \beta}{2} \leq \sqrt{-E_j(\beta)} \leq \frac{\beta + \ln \beta}{2}$$

as the coupling parameter  $\beta \rightarrow +\infty$ .

**Proof.** The upper bound follows from (9) and Proposition 2. To prove the other inequality, note that one can construct a  $C^4$  loop  $\tilde{\gamma}$  such that

$\gamma \subset \tilde{\gamma}$ . Denote by  $N_\beta$  the self-adjoint operator in  $L^2(\mathbb{R}^2)$  associated with the sesquilinear form

$$n_\beta(f, f) = \int_{\mathbb{R}^2} |\nabla f|^2 dx - \beta \int_{\tilde{\gamma}} |f|^2 dS$$

and denote by  $\tilde{E}_j(\beta)$  its eigenvalues arranged in the ascending order with their multiplicities taken into account. By the max-min principle, we have  $\tilde{E}_j(\beta) \leq E_j(\beta)$  where the left-hand side behaves by [6] asymptotically as

$$\tilde{E}_j(\beta) = -\frac{\beta^2}{4} + \kappa_j + \mathcal{O}\left(\frac{\log \beta}{\beta}\right),$$

$\tilde{\kappa}_j$  being the eigenvalues of the Schrödinger operator with the curvature-induced potential on  $\tilde{\gamma}$ . This gives  $\sqrt{-E_j(\beta)} \geq \frac{\beta}{2}(1 + \mathcal{O}(\beta^{-2}))$ , and thus the sought result.  $\square$

Let  $u_{j,\beta}$  be now an  $L^2$ -normalized eigenfunction of  $H_\beta$  corresponding to the eigenvalue  $E_j(\beta)$ ,  $j \in \mathbb{N}$ . By [3, 10] one can represent it as

$$u_{j,\beta}(x) = \int_{\gamma} G_0(x, y; E) F_{j,\beta}(y) dS_y, \quad (20)$$

where  $F_{j,\beta} \in L^2(\gamma)$  is an appropriate solution to the integral equation

$$\int_{\gamma} G_0(x, y; E_j(\beta)) F_{j,\beta}(y) dS_y = \frac{1}{\beta} F_{j,\beta}(x), \quad x \in \gamma, \quad (21)$$

coming from the corresponding Krein's formula, and  $G_0$  is the Green function of the two-dimensional free Laplacian given explicitly by

$$G_0(x, y; z) = \frac{1}{2\pi} K_0(\sqrt{-z}|x - y|);$$

here and in the following  $K_\nu$  denotes the modified Bessel function of order  $\nu$ , see [1, Section 9.6].

The following estimate will be of crucial importance for our result.

**Lemma 11.**  $\|F_{j,\beta}\|_{L^2(\gamma)} = \mathcal{O}(\beta^2 \sqrt{\log \beta})$  holds as  $\beta \rightarrow +\infty$ .

**Proof.** Throughout the proof again  $C_j$  will denote various positive constants. To avoid using cumbersome notation we identify the function  $F_{j,\beta}(\cdot)$  with  $F_{j,\beta}(\Phi(\cdot, 0)) \equiv F_{j,\beta}(\Gamma(\cdot))$  and write simply  $E$  instead of  $E_j(\beta)$ .

We will employ the following well-know relation [1, Eqs. 9.7.2 and 9.6.27]:

$$K_\nu(w) = \sqrt{\frac{\pi}{2w}} e^{-w} (1 + o(1)), \quad w \rightarrow +\infty, \quad \nu = 0, 1, \quad (22)$$

$$K'_0 = -K_1. \quad (23)$$

According to (20) and (21), one has

$$u_{j,\beta}|_{\gamma} = \frac{1}{\beta} F_{j,\beta}, \quad (24)$$

and moreover, using (20) and (23) we can write

$$\nabla u_{j,\beta}(x) = \frac{1}{2\pi} \int_{\gamma} \frac{\sqrt{-E}(y-x)}{|x-y|} K_1(\sqrt{-E}|x-y|) F_{j,\beta}(y) dS_y.$$

Another property to use [1, Eqs. 9.6.10 and 9.6.11] is the representation

$$K_1(t) = \frac{1}{t} + M(t), \quad M(t) = tg_1(t) \log t + g_2(t), \quad (25)$$

where  $g_1$  and  $g_2$  are analytic functions. It yields

$$\begin{aligned} \nabla u_{j,\beta}(x) &= \frac{1}{2\pi} \int_{\gamma} \frac{(y-x)}{|x-y|^2} F_{j,\beta}(y) dS_y \\ &\quad + \frac{1}{2\pi} \int_{\gamma} \frac{\sqrt{-E}(y-x)}{|x-y|} M(\sqrt{-E}|y-x|) F_{j,\beta}(y) dS_y. \end{aligned} \quad (26)$$

Let us estimate the expression  $n(s) \cdot \nabla u_{j,\beta}(\Phi(s, t))$ . In view of the representation (25) and the asymptotics (22) we have a uniform bound  $|M(w)| \leq 2\pi C_1$  for all  $w > 0$ , and therefore

$$\begin{aligned} &\left| n(s) \cdot \nabla u_{j,\beta}(\Phi(s, t)) \right| \\ &\leq \left| \frac{1}{2\pi} \int_0^L \frac{n(s) \cdot (\Phi(\sigma, 0) - \Phi(s, t))}{|\Phi(\sigma, 0) - \Phi(s, t)|^2} F_{j,\beta}(\sigma) d\sigma \right| + C_1 \sqrt{-E} \|F_{j,\beta}\|_{L^1(\gamma)}. \end{aligned} \quad (27)$$

Furthermore, for large enough  $\beta$  Lemma 4 implies the estimate

$$\frac{1}{2\pi |\Phi(\sigma, 0) - \Phi(s, t)|^2} \leq \frac{C_4}{(s - \sigma)^2 + t^2}.$$

for all  $s, \sigma \in (0, L)$  and  $t \in (-a, a)$ , recall the assumption (5). Next note that  $\Phi(\sigma, 0) - \Phi(s, t) = -tn(s) + \Gamma(\sigma) - \Gamma(s)$ , hence using (11) we get

$$n(s) \cdot (\Phi(\sigma, 0) - \Phi(s, t)) = -t + (\sigma - s)^2 \rho(\sigma, s),$$

where  $\rho(\sigma, s) = n(s) \cdot \rho_1(\sigma, s)$  is uniformly bounded on  $[0, L] \times [0, L]$ . Consequently, there are  $C_5, C_6 > 0$  such that

$$\left| \frac{n(s) \cdot (\Phi(\sigma, 0) - \Phi(s, t))}{|\Phi(\sigma, 0) - \Phi(s, t)|^2} \right| \leq C_5 \frac{|t| + C_6(s - \sigma)^2}{(s - \sigma)^2 + t^2} \leq C_5 \frac{|t|}{(s - \sigma)^2 + t^2} + C_5 C_6$$

and

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^L \frac{n(s) \cdot (\Phi(\sigma, 0) - \Phi(s, t))}{|\Phi(\sigma, 0) - \Phi(s, t)|^2} F_{j,\beta}(\sigma) d\sigma \right| \\ \leq C_5 \int_0^L \frac{|t|}{(s - \sigma)^2 + t^2} |F_{j,\beta}(\sigma)| d\sigma + C_5 C_6 \|F_{j,\beta}\|_{L^1(\gamma)}. \end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_0^L \frac{|t|}{(s - \sigma)^2 + t^2} |F_{j,\beta}(\sigma)| d\sigma &\leq \left( \int_0^L \frac{|t|^2}{((s - \sigma)^2 + t^2)^2} d\sigma \right)^{1/2} \|F_{j,\beta}\|_{L^2(\gamma)} \\ &\leq \left( \int_{\mathbb{R}} \frac{|t|^2}{((s - \sigma)^2 + t^2)^2} d\sigma \right)^{1/2} \|F_{j,\beta}\|_{L^2(\gamma)} \\ &= |t| \left( \int_{\mathbb{R}} \frac{d\sigma}{(\sigma^2 + t^2)^2} \right)^{1/2} \|F_{j,\beta}\|_{L^2(\gamma)} \\ &= |t|^{-1/2} \left( \int_{\mathbb{R}} \frac{d\xi}{(\xi^2 + 1)^2} \right)^{1/2} \|F_{j,\beta}\|_{L^2(\gamma)} = C_7 |t|^{-1/2} \|F_{j,\beta}\|_{L^2(\gamma)}. \quad (28) \end{aligned}$$

Putting everything together and using a rough estimate  $E = \mathcal{O}(\beta)$  from Lemma 10, we get a bound

$$\left| n(s) \cdot \nabla u_{j,\beta}(\Phi(s, t)) \right| \leq C_8 (|t|^{-1/2} + \beta) \|F_{j,\beta}\|_{L^2(\gamma)} \quad (29)$$

with some constant  $C_8 > 0$ . Next we denote

$$\delta := \frac{1}{\beta^2 \log \beta}$$

and for  $\beta$  large enough we construct a new function  $v$  on  $\Omega(\delta)$  by

$$v_{j,\beta}(\Phi(s, t)) := u_{j,\beta}(\Phi(s, 0)), \quad (s, t) \in (0, L) \times (-\delta, \delta),$$

for which the triangle inequality yields

$$\|u\|_{L^2(\Omega(\delta))} \geq \|v_{j,\beta}\|_{L^2(\Omega(\delta))} - \|u_{j,\beta} - v_{j,\beta}\|_{L^2(\Omega(\delta))}. \quad (30)$$

Using (24), one can write now the following estimates:

$$\begin{aligned} \|v_{j,\beta}\|_{L^2(\Omega(\delta))}^2 &\geq C_9 \int_{-\delta}^{\delta} \int_0^L |u_{j,\beta}(\Phi(s, 0))|^2 ds dt \\ &= \frac{C_9}{\beta^2} \int_{-\delta}^{\delta} \int_0^L |F_{j,\beta}(s)|^2 ds dt = \frac{C_{10}^2 \delta}{\beta^2} \|F_{j,\beta}\|_{L^2(\gamma)}^2 = \frac{C_{10}^2}{\beta^4 \log \beta} \|F_{j,\beta}\|_{L^2(\gamma)}^2. \quad (31) \end{aligned}$$

On the other hand, the second term on the right-hand side of (30) satisfies

$$\|u_{j,\beta} - v_{j,\beta}\|_{L^2(\Omega(\delta))}^2 \leq C_{11} \int_0^L \int_{-\delta}^{\delta} \left| u_{j,\beta}(\Phi(s, t)) - u_{j,\beta}(\Phi(s, 0)) \right|^2 dt ds. \quad (32)$$

To estimate the integrated function, we employ the relation

$$\frac{d}{dt} u_{j,\beta}(\Phi(s, t)) = n(s) \cdot \nabla u_{j,\beta}(\Phi(s, t)),$$

which yields, through (29), the bound

$$\begin{aligned} & \left| u_{j,\beta}(\Phi(s, t)) - u_{j,\beta}(\Phi(s, 0)) \right| \\ &= \left| \int_0^t n(s) \cdot \nabla u_{j,\beta}(\Phi(s, \xi)) d\xi \right| \leq \int_0^{|t|} \left| n(s) \cdot \nabla u_{j,\beta}(\Phi(s, \xi)) \right| d\xi \\ &\leq C_8 \int_0^{|t|} (|\xi|^{-1/2} + \beta) d\xi \|F_{j,\beta}\|_{L^2(\gamma)} = C_8 (2|t|^{1/2} + |t|\beta) \cdot \|F_{j,\beta}\|_{L^2(\gamma)}, \end{aligned}$$

and consequently,

$$\begin{aligned} \|u_{j,\beta} - v_{j,\beta}\|_{L^2(\Omega(\delta))}^2 &\leq 8C_8 C_{11} \int_0^L \int_{-\delta}^{\delta} (|t| + \beta^2 t^2) dt ds \|F_{j,\beta}\|_{L^2(\gamma)}^2 \\ &\leq C_{12}(\delta^2 + \delta^3 \beta^2) \|F_{j,\beta}\|_{L^2(\gamma)}^2 \leq \frac{C_{13}^2}{\beta^4 \log^2 \beta} \|F_{j,\beta}\|_{L^2(\gamma)}^2. \end{aligned} \quad (33)$$

Substituting finally (31) and (33) into (30) we obtain

$$\begin{aligned} 1 &= \|u_{j,\beta}\|_{L^2(\mathbb{R}^2)} \geq \|u_{j,\beta}\|_{L^2(\Omega(\delta))} \\ &\geq \left( \frac{C_{10}}{\beta^2 \sqrt{\log \beta}} - \frac{C_{13}}{\beta^2 \log \beta} \right) \|F_{j,\beta}\|_{L^2(\gamma)} \geq \frac{C_{14}}{\beta^2 \sqrt{\log \beta}} \|F_{j,\beta}\|_{L^2(\gamma)}, \end{aligned}$$

which gives the sought result.  $\square$

**Lemma 12.** *For any  $k, c > 0$  one can find a  $D > 0$  such that*

$$|u_{j,\beta}(x)| \leq D\beta^2 \exp\left(-\frac{(\beta - \log \beta)d(x, \gamma)}{2}\right), \quad (34)$$

$$|\nabla u_{j,\beta}(x)| \leq D\beta^3 \exp\left(-\frac{(\beta - \log \beta)d(x, \gamma)}{2}\right) \quad (35)$$

*holds whenever  $x \notin W\left(\frac{k \log \beta - c}{\beta}\right)$ .*

**Proof.** Recall that we have the integral representation (20) for the eigenfunction  $u_{j,\beta}$ , hence using Lemma 11 and Cauchy-Schwarz inequality we infer that

$$\begin{aligned} |u_{j,\beta}(x)| &\leq \sup_{y \in \gamma} \left| K_0(\sqrt{-E_j(\beta)}|x-y|) \right| \cdot \|F_{j,\beta}\|_{L^1(\gamma)} \\ &\leq C_1 \beta^2 \sqrt{\log \beta} \sup_{y \in \gamma} \left| K_0(\sqrt{-E_j(\beta)}|x-y|) \right|. \end{aligned} \quad (36)$$

For  $x \notin W\left(\frac{k \log \beta - c}{\beta}\right)$  and  $y \in \gamma$  we have, using Lemma 10,

$$\begin{aligned} \sqrt{-E_j(\beta)}|x-y| &\geq \sqrt{-E_j(\beta)} d(x, \gamma) \\ &\geq \frac{\beta - \log \beta}{2} \frac{k \log \beta - c}{\beta} = \frac{k \log \beta}{2} + \mathcal{O}(1) \end{aligned}$$

as  $\beta \rightarrow +\infty$ . For fixed  $x, y$  the asymptotics (22) and Lemma 10 imply

$$\begin{aligned} &\left| K_0(\sqrt{-E_j(\beta)}|x-y|) \right| \\ &\leq C_3 \left( \sqrt{-E_j(\beta)}|x-y| \right)^{-1/2} \exp \left( -\frac{(\beta - \log \beta)|x-y|}{2} \right) \\ &\leq C_3 \left( \sqrt{-E_j(\beta)}d(x, \gamma) \right)^{-1/2} \exp \left( -\frac{(\beta - \log \beta)d(x, \gamma)}{2} \right) \\ &\leq \frac{C_4}{\sqrt{\log \beta}} \exp \left( -\frac{(\beta - \log \beta)d(x, \gamma)}{2} \right). \end{aligned}$$

Combining this inequality with (36) we obtain the bound (34). To estimate  $\nabla u_{j,\beta}$  we use (23) and write

$$\nabla u_{j,\beta}(x) = -\sqrt{-E_j(\beta)} \int_{\gamma} \nabla_x |x-y| K_1 \left( \sqrt{-E_j(\beta)} |x-y| \right) F_{j,\beta}(y) dS_y.$$

It is now enough note that  $|\nabla_x |x-y|| \leq 1$  and that  $E_j(\beta) = \mathcal{O}(\beta)$  by Lemma 10, hence estimating the integral again with the help of (22) we arrive at the bound (35).  $\square$

## 6 Cut-off functions

In this section we introduce a family of cut-off functions that will be used in the following when we will apply the max-min principle in the last step of the argument. An inspiration for this type of constructions came from the paper [9].



We choose a mollifying function  $C^\infty$  function  $\psi : \mathbb{R} \rightarrow [0, +\infty)$  such that

$$\psi(s) = 1 \text{ for } s \geq 0 \text{ and } \psi(s) = 0 \text{ for } s \leq -1.$$

Next we consider the function  $\rho_a : P(a) \rightarrow \mathbb{R}$ ,

$$\rho_a(s, t) = \min \{|a - t|, |a + t|, L + a - s, s + a\},$$

in other words,  $\rho_a(s, t)$  is the distance between the point  $(s, t) \in P(a)$  and the boundary of the rectangle  $P(a)$ . We use it to introduce the function  $R_\beta : \Pi(a) \rightarrow \mathbb{R}$  by

$$R_\beta(x) = \rho_a(\Phi^{(-1)}(x)).$$

where  $\Phi^{(-1)}(x)$  means the pre-image of the point  $x \in \Pi(a)$  with respect to the map (4) and the parameters are related by (5). Finally, for sufficiently large  $\beta$  and we introduce the function  $g_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$g_\beta(x) = \begin{cases} \psi\left(\frac{\log R_\beta(x) + \log \beta}{\log \log \beta}\right), & x \in \Pi(a), \\ 0, & \text{otherwise.} \end{cases} \quad (37)$$

Note that  $g_\beta$  belongs to  $H^1(\mathbb{R}^2)$  and has a compact support since  $g(x) = 0$  for all  $x \notin \Pi(a)$ . In addition, we have  $g(x) = 0$  for those  $x \in \Pi(a)$  that can be represented as  $x = \Phi(s, t)$  with  $\rho_a(s, t) \leq \frac{1}{\beta \log \beta}$ . On the other hand,  $g(x) = 1$  holds for  $x \in \Pi(a)$  with  $R_\beta(x) \geq \frac{1}{\beta}$ . In particular,

$$\text{supp } \nabla g_\beta \subset \Theta(\beta) := \left\{ \Phi(s, t) : (s, t) \in P(a), \quad \frac{1}{\beta \log \beta} \leq \rho_a(s, t) \leq \frac{1}{\beta} \right\},$$

$$\text{supp}(1 - g) \subset V(\beta) := \left\{ \Phi(s, t) : (s, t) \in P(a), \quad \rho_a(s, t) \leq \frac{1}{\beta} \right\}.$$

**Lemma 13.** *In the limit  $\beta \rightarrow +\infty$  one has*

$$\begin{aligned} \iint_{\Theta(\beta)} |\nabla g_\beta(x)| dx &= \mathcal{O}(1), \\ \iint_{\Theta(\beta)} |\nabla g_\beta(x)|^2 dx &= \mathcal{O}(\beta \log \beta). \end{aligned}$$

**Proof.** Let  $D_{s,t}\Phi$  denote the Jacobian matrix value of the map  $\Phi$  at  $(s, t)$ . We have

$$\begin{aligned} \nabla g_\beta(\Phi(s, t)) &= \psi' \left( \frac{\log R_\beta(\Phi(s, t)) + \log \beta}{\log \log \beta} \right) \frac{\nabla R_\beta(\Phi(s, t))}{R_\beta(\Phi(s, t)) \log \log \beta} \\ &= \psi' \left( \frac{\log \rho_a(s, t) + \log \beta}{\log \log \beta} \right) \frac{1}{\rho_a(s, t) \log \log \beta} \nabla \rho_a(s, t) (D_{s,t}\Phi)^{-1}. \end{aligned}$$

We have  $|\nabla \rho_a(s, t)| \leq 1$  and  $\|(D_{s,t}\Phi)^{-1}\| \leq M$  for some  $M > 0$  and all  $(s, t) \in P(a)$  if  $\beta$  is sufficiently large. Hence it holds

$$|\nabla g_\beta(\Phi(s, t))| \leq \frac{C_1}{\rho_a(s, t) \log \log \beta}.$$

with some  $C_1 > 0$ , and

$$\iint_{\Theta(\beta)} |\nabla g_\beta(x)|^\nu dx \leq \frac{C_1^\nu C_2}{(\log \log \beta)^\nu} \iint_{\substack{(s,t) \in P(a), \\ \frac{1}{\beta \log \beta} \leq \rho_a(s,t) \leq \frac{1}{\beta}}} \frac{ds dt}{\rho_a(s, t)^\nu}, \quad \nu = 1, 2. \quad (38)$$

Since the integration variables run through the set

$$\frac{1}{\beta \log \beta} \leq \rho_a(s, t) \leq \frac{1}{\beta},$$

the integral on the right-hand side is the sum of contributions from integration over four rectangles and eight triangles. Using the obvious symmetries, we can rewrite it as

$$\begin{aligned} I_\nu(\beta) &:= \iint_{\substack{(s,t) \in P(a) \\ \frac{1}{\beta \log \beta} < \rho_a(s,t) < \frac{1}{\beta}}} \frac{ds dt}{\rho_a(s, t)^\nu} = 2 \int_{-a+\frac{1}{\beta}}^{L+a-\frac{1}{\beta}} \int_{\frac{1}{\beta \log \beta}}^{\frac{1}{\beta}} \frac{1}{t^\nu} dt ds \\ &\quad + 2 \int_{\frac{1}{\beta \log \beta}}^{\frac{1}{\beta}} \int_{-a+\frac{1}{\beta}}^{a-\frac{1}{\beta}} \frac{1}{s^\nu} dt ds + 8 \int_{\frac{1}{\beta \log \beta}}^{\frac{1}{\beta}} \int_{\frac{1}{\beta \log \beta}}^s \frac{1}{t^\nu} dt ds \leq C_3 \int_{\frac{1}{\beta \log \beta}}^{\frac{1}{\beta}} \frac{dt}{t^\nu}. \end{aligned}$$

for  $\nu = 1, 2$  and some  $C_3 > 0$ . Hence

$$I_1(\beta) \leq C_3 \int_{\frac{1}{\beta \log \beta}}^{\frac{1}{\beta}} \frac{dt}{t} = C_3 \log \log \beta.$$

and

$$I_2(\nu) = C_3 \int_{\frac{1}{\beta \log \beta}}^{\frac{1}{\beta}} \frac{dt}{t^2} = C_3(\beta \log \beta - \beta).$$

Finally, by (38) we infer that

$$\iint_{\Theta(\beta)} |\nabla g_\beta(x)| dx = \frac{C_1 C_2 I_1(\beta)}{\log \log \beta} \leq \frac{C_1 C_2 C_3 \log \log \beta}{\log \log \beta} = \mathcal{O}(1)$$

and

$$\iint_{\Theta(\beta)} |\nabla g_\beta(x)|^2 dx = \frac{C_1^2 C_2 I_2(\beta)}{(\log \log \beta)^2} \leq \frac{C_1^2 C_2 C_3 (\beta \log \beta - \beta)}{(\log \log \beta)^2} = \mathcal{O}(\beta \log \beta).$$

holds as  $\beta \rightarrow +\infty$  which we have set out to prove.  $\square$

**Lemma 14.** *For sufficiently large  $\beta$  there is a constant  $D > 0$  such that*

$$|u_{j,\beta}(x)| \leq \frac{D}{\beta}, \quad (39)$$

$$|\nabla u_{j,\beta}(x)| \leq D \quad (40)$$

holds for all  $x \in V(\beta)$ .

**Proof.** By Corollary 8 there exists a  $C_1 > 0$  such that

$$d(x, \gamma) \geq a - C_1 a^2 \text{ for all } x \in \partial\Pi(a),$$

holds provided  $\beta$  is sufficiently large. On the other hand, for any  $x = \Phi(s, t) \in V(\beta)$  one can find  $(s', t') \in \partial P(a)$  with

$$\rho_a(s, t) = \sqrt{(s - s')^2 + (t - t')^2} \leq \frac{1}{\beta}.$$

As  $\partial\Pi(a) = \Phi(\partial\Pi(a))$ , it follows from Lemma 4 that for all  $x \in V(\beta)$

$$d(x, \partial\Pi(a)) \leq |\Phi(s, t) - \Phi(s', t')| \leq C_2 \sqrt{(s - s')^2 + (t - t')^2} \leq \frac{C_2}{\beta}$$

holds with some  $C_2 > 0$ . Consequently, for sufficiently large  $\beta$  we have

$$V(\beta) \subset \mathbb{R}^2 \setminus W\left(a - \frac{2C_2}{\beta}\right) = \mathbb{R}^2 \setminus W\left(\frac{6 \log \beta - 2C_2}{\beta}\right),$$

and Lemma 12 is applicable. For  $x \in V(\beta)$  and large  $\beta$  we can estimate

$$\sqrt{-E_j(\beta)} d(x, \gamma) \geq \frac{\beta - \log \beta}{2} \left( \frac{6 \log \beta}{\beta} - \frac{2C_2}{\beta} \right) = 3 \log \beta + \mathcal{O}(1),$$

by Lemma 10, hence applying (34) and (35) we get the sought bounds.  $\square$

## 7 Using the max-min principle

Let us fix now an integer  $N \geq 1$ . Consider the first  $N$  eigenvalues  $E_j(\beta)$  and the associated *orthonormal* eigenfunctions  $u_{j,\beta}$  of  $H_\beta$  and denote

$$\varphi_{j,\beta} := g_\beta u_{j,\beta},$$

where  $g_\beta$  is the function (37). As  $\text{supp } g_\beta \subset \Pi(a)$ , one has  $\varphi_{j,\beta} \in H_0^1(\Pi(a))$ . Following the usual convention, we denote here and in the following by  $\delta_{jl}$  the Kronecker delta symbol.

**Lemma 15.** *In the limit  $\beta \rightarrow +\infty$  one has*

$$\langle \varphi_{j,\beta}, \varphi_{l,\beta} \rangle_{L^2(\Pi(a))} = \delta_{jl} + \mathcal{O}(\beta^{-2}). \quad (41)$$

**Proof.** Denote for brevity  $S_\beta := W\left(\frac{5 \log \beta}{\beta}\right)$ . In a way similar to the proof of Lemma 14 one can show that for all sufficiently large  $\beta$  we have  $S_\beta \subset \Pi(a)$  and  $g_\beta|_{S_\beta} = 1$ . Moreover, for  $x \notin S_\beta$  one can estimate  $u_{j,\beta}(x)$  with the help of Lemma 12. Hence using first the boundedness of the function  $g_\beta$  and applying subsequently Lemma 9, we get

$$\begin{aligned} & \left| \langle u_{j,\beta}, u_{l,\beta} \rangle_{L^2(\mathbb{R}^2)} - \langle \varphi_{j,\beta}, \varphi_{l,\beta} \rangle_{L^2(\Pi(a))} \right| \\ &= \left| \langle u_{j,\beta}, u_{l,\beta} \rangle_{L^2(\mathbb{R}^2)} - \langle \varphi_{j,\beta}, \varphi_{l,\beta} \rangle_{L^2(\mathbb{R}^2)} \right| \\ &= \left| \iint_{\mathbb{R}^2} (1 - g_\beta(x)^2) \overline{u_{j,\beta}(x)} u_{l,\beta}(x) dx \right| = \left| \iint_{\mathbb{R}^2 \setminus S_\beta} (1 - g_\beta(x)^2) \overline{u_{j,\beta}(x)} u_{l,\beta}(x) dx \right| \\ &\leq C_1 \iint_{\mathbb{R}^2 \setminus S_\beta} |\overline{u_{j,\beta}(x)} u_{l,\beta}(x)| dx \leq C_2 \beta^4 \iint_{\mathbb{R}^2 \setminus S_\beta} e^{-(\beta - \log \beta)d(x,\gamma)} dx = \mathcal{O}(\beta^{-2}) \end{aligned}$$

with some constants  $C_1, C_2 > 0$ . As  $\{u_{j,\beta}\}$  is an orthonormal system by assumption, we arrive at the relation (41).  $\square$

**Lemma 16.** *In the limit  $\beta \rightarrow +\infty$  one has*

$$\langle \nabla u_{j,\beta}, \nabla u_{l,\beta} \rangle_{L^2(\Pi(a))} - \beta \int_{\gamma} \overline{u_{j,\beta}(s)} u_{l,\beta}(s) dS = E_j(\beta) \delta_{jl} + \mathcal{O}(\beta^{-1}).$$

**Proof.** Note first that the relations

$$\langle \nabla u_{j,\beta}, \nabla u_{l,\beta} \rangle_{L^2(\mathbb{R}^2)} - \beta \int_{\gamma} \overline{u_{j,\beta}(s)} u_{l,\beta}(s) dS = E_j(\beta) \delta_{jl}$$

hold by assumption and that a certain neighborhood of  $\gamma$  is included into  $\Pi(a)$ , hence it is sufficient to check the estimate

$$\langle \nabla u_{j,\beta}, \nabla u_{l,\beta} \rangle_{L^2(\mathbb{R}^2 \setminus \Pi(a))} = \mathcal{O}(\beta^{-1}).$$

As in the proof of Lemma 14 we can check that the inclusion

$$W\left(\frac{6 \log \beta - C_1}{\beta}\right) \subset \Pi(a).$$

holds for some  $C_1 > 0$  and all sufficiently large  $\beta$ . Using then the estimate (35) and subsequently Lemma 9, we get

$$\begin{aligned} \left| \langle \nabla u_{j,\beta}, \nabla u_{l,\beta} \rangle_{L^2(\mathbb{R}^2 \setminus \Pi(a))} \right| &\leq \iint_{\mathbb{R}^2 \setminus W\left(\frac{6 \log \beta - C_1}{\beta}\right)} |\nabla u_{j,\beta}(x)| \cdot |\nabla u_{l,\beta}(x)| dx \\ &\leq C_2 \beta^6 \iint_{\mathbb{R}^2 \setminus W\left(\frac{6 \log \beta - C_1}{\beta}\right)} e^{-(\beta - \log \beta)d(x,\gamma)} dx \leq C_3 \frac{\beta^6}{\beta^7} = \mathcal{O}\left(\frac{1}{\beta}\right). \end{aligned}$$

□

Our principal estimate concerns the question what happens if  $u_{j,\beta}$  in the above formula is replaced by the mollified function; our aim is to show that this makes the error term worse but only by a logarithmic factor.

**Lemma 17.** *In the limit  $\beta \rightarrow +\infty$  one has*

$$\langle \nabla \varphi_{j,\beta}, \nabla \varphi_{l,\beta} \rangle_{L^2(\Pi(a))} - \beta \int_{\gamma} \overline{\varphi_{j,\beta}(s)} \varphi_{l,\beta}(s) dS = E_j(\beta) \delta_{jl} + \mathcal{O}\left(\frac{\log \beta}{\beta}\right).$$

**Proof.** Using  $\varphi_{j,\beta}|_{\gamma} = u_{j,\beta}|_{\gamma}$  let us write the expression in question as

$$\begin{aligned} \langle \nabla \varphi_{j,\beta}, \nabla \varphi_{l,\beta} \rangle_{L^2(\Pi(a))} - \beta \int_{\gamma} \overline{\varphi_{j,\beta}(s)} \varphi_{l,\beta}(s) dS \\ = \langle \nabla u_{j,\beta}, \nabla u_{l,\beta} \rangle_{L^2(\Pi(a))} - \beta \int_{\gamma} \overline{u_{j,\beta}(s)} u_{l,\beta}(s) dS \\ + \iint_{\Pi(a)} (g_{\beta}(x)^2 - 1) \overline{\nabla u_{j,\beta}(x)} \cdot \nabla u_{l,\beta}(x) dx \\ + \iint_{\Pi(a)} |\nabla g_{\beta}(x)|^2 \overline{u_{j,\beta}(x)} u_{l,\beta}(x) dx \\ + \iint_{\Pi(a)} g_{\beta}(x) \overline{u_{j,\beta}(x)} \nabla g_{\beta}(x) \cdot \nabla u_{l,\beta}(x) dx \\ + \iint_{\Pi(a)} g_{\beta}(x) u_{l,\beta}(x) \overline{\nabla u_{j,\beta}(x)} \cdot \nabla g_{\beta}(x) dx. \end{aligned}$$

The sum of the first two terms on the right-hand side has been already estimated in Lemma 16, hence we just need to show that the sum of the last

four terms on the right-hand side is of order  $\mathcal{O}(\beta^{-1} \log \beta)$ . By definition of the function  $g_\beta$  and Lemma 14 there are constants  $C_1, C_2 > 0$  such that

$$\begin{aligned} I_1 &:= \left| \iint_{\Pi(a)} (g_\beta(x)^2 - 1) \overline{\nabla u_{j,\beta}(x)} \nabla u_{l,\beta}(x) dx \right| \\ &= \left| \iint_{V(\beta)} (g_\beta(x)^2 - 1) \overline{\nabla u_{j,\beta}(x)} \nabla u_{l,\beta}(x) dx \right| \\ &\leq C_1 \iint_{V(\beta)} \left| \overline{\nabla u_{j,\beta}(x)} \nabla u_{l,\beta}(x) \right| dx \leq C_2 |V(\beta)|. \end{aligned}$$

Furthermore, by definition of  $V(\beta)$  we have

$$V(\beta) = \Phi(U), \quad U = \left\{ (s, t) \in P(a) : \rho_a(s, t) \leq \frac{1}{\beta} \right\},$$

and since the measure  $|U|$  is of order  $\mathcal{O}(\beta^{-1})$ , we get also  $|V(\beta)| = \mathcal{O}(\beta^{-1})$ , which in turn gives  $I_1 = \mathcal{O}(\beta^{-1})$ .

Using next the inclusion  $\text{supp } \nabla g_\beta \subset \Theta(\beta) \subset V(\beta)$ , Lemma 14 and after that Lemma 13, we have

$$\begin{aligned} I_2 &:= \left| \iint_{\Pi(a)} |\nabla g_\beta(x)|^2 \overline{u_{j,\beta}(x)} u_{l,\beta}(x) dx \right| \\ &= \left| \iint_{\Theta(\beta)} |\nabla g_\beta(x)|^2 \overline{u_{j,\beta}(x)} u_{l,\beta}(x) dx \right| \\ &\leq \frac{C_3}{\beta^2} \iint_{\Theta(\beta)} \left| \nabla g_\beta(x) \right|^2 dx = \mathcal{O}\left(\frac{\log \beta}{\beta}\right). \end{aligned}$$

Using the same reasoning we infer that

$$\begin{aligned} I_{j,l} &:= \left| \iint_{\Pi(a)} g_\beta(x) \overline{u_{j,\beta}(x)} \nabla g_\beta(x) \nabla u_{l,\beta}(x) dx \right| \\ &= \left| \iint_{\Theta(\beta)} g_\beta(x) \overline{u_{j,\beta}(x)} \nabla g_\beta(x) \nabla u_{l,\beta}(x) dx \right| \\ &\leq \frac{C_4}{\beta} \iint_{\Theta(\beta)} \left| \nabla g_\beta(x) \right| dx = \mathcal{O}\left(\frac{1}{\beta}\right). \end{aligned}$$

Putting the estimates together we find

$$I_1 + I_2 + I_{j,l} + I_{l,j} = \mathcal{O}\left(\frac{\log \beta}{\beta}\right),$$

which concludes the proof.  $\square$

Now we are in position to complete the proof of our main result.

*Proof of Proposition 3.* Fix an integer  $N \geq 1$ . By the max-min principle one has

$$\Lambda_N(\beta) = \max_{G \in S_N} \min_{0 \neq f \in G} \frac{\iint_{\Pi(a)} |\nabla f|^2 dx - \beta \int_{\gamma} |f|^2 dS}{\|f\|_{L^2(\Pi(a))}^2},$$

where  $S_N$  stands for the family of the subspaces of  $H_0^1(\Pi(a))$  the codimension of which in  $L^2(\Pi(a))$  equals  $N - 1$ . In view of Lemma 15, the functions  $\varphi_{j,\beta}$ ,  $j = 1, \dots, N$ , are linearly independent in  $L^2(\Pi(a))$  for all sufficiently large  $\beta$ , hence each subspace  $G \in S_N$  contains at least one linear combination  $\varphi$  of the form

$$\varphi = \sum_{j=1}^N b_j \varphi_{j,\beta}, \quad b = (b_1, \dots, b_N) \in \mathbb{C}^N, \quad \|b\|_{\mathbb{C}^N} = 1.$$

Using once more Lemma 15, we find that

$$\|\varphi\|_{L^2(\Pi(a))}^2 \geq 1 - \frac{C_1}{\beta^2}.$$

holds for large  $\beta$  with a constant  $C_1 > 0$ . On the other hand, Lemma 17 yields

$$\begin{aligned} & \iint_{\Pi(a)} |\nabla \varphi|^2 dx - \beta \int_{\gamma} |\varphi|^2 dS \\ &= \sum_{j,l=1}^N \bar{b}_j b_l \left( \langle \nabla \varphi_{j,\beta}, \nabla \varphi_{l,\beta} \rangle_{L^2(\Pi(a))} - \beta \int_{\gamma} \overline{\varphi_{j,\beta}(s)} \varphi_{l,\beta}(s) dS \right) \\ &= \sum_{j,l=1}^N \bar{b}_j b_l \left( E_j(\beta) \delta_{jl} + O\left(\frac{\log \beta}{\beta}\right) \right) = \sum_{j=1}^N E_j(\beta) |b_j|^2 + O\left(\frac{\log \beta}{\beta}\right) \\ &\leq E_N(\beta) + O\left(\frac{\log \beta}{\beta}\right). \quad (42) \end{aligned}$$

Using the above estimates, we conclude that there are  $C_2, C_3 > 0$  such that

$$\begin{aligned} \min_{0 \neq f \in G} \frac{\iint_{\Pi(a)} |\nabla f|^2 dx - \beta \int_{\gamma} |f|^2 dS}{\|f\|_{L^2(\Pi(a))}^2} &\leq \frac{\iint_{\Pi(a)} |\nabla \varphi|^2 dx - \beta \int_{\gamma} |\varphi|^2 dS}{\|\varphi\|_{L^2(\Pi(a))}^2} \\ &\leq \frac{E_N(\beta) + C_2 \frac{\log \beta}{\beta}}{1 - C_1 \beta^{-2}} \leq E_N(\beta) + C_3 \frac{\log \beta}{\beta}. \end{aligned}$$

What is important is that the constant  $C_3$  can be chosen independent of the vector  $b$  and hence independent of  $G \in S_N$ , then we have automatically

$$\Lambda_N(\beta) \leq E_N(\beta) + C_3 \frac{\log \beta}{\beta}.$$

Combining this with (9) we obtain  $\Lambda_N(\beta) - E_N(\beta) = \mathcal{O}(\beta^{-1} \log \beta)$ .  $\square$

## 8 Acknowledgmenets

The second named author thanks the Doppler Institute in Prague for the warm hospitality during the stay in May-June 2012. The research was partially supported by ANR NOSEVOL and GDR Dynamique Quantique, and by Czech Science Foundation within the project P203/11/0701.

## References

- [1] M. Abramowitz, I. A. Stegun (eds.): Handbook of mathematical functions with formulas, graphs, and mathematical tables. 10th printing (volume 55 of Applied Mathematics Series, US National Bureau of Standards, 1972). Available online at <http://www.math.sfu.ca/~cbm/aands/>.
- [2] M. S. Agranovich: *Strongly elliptic second-order systems with boundary vonditions on a nonclosed Lipschitz surface*. Funct. Anal. Appl. **45**:1 (2011), 1–12.
- [3] J. F. Brasche, P. Exner, Yu. A. Kuperin, P. Šeba: *Schrödinger operators with singular ineractions*. J. Math. Anal. Appl. **184** (1994), 112–139.
- [4] M. Dauge, B. Helffer: *Eigenvalues variation I. Neumann problem for Sturm-Liouville operators*. J. Differential Eqs **104** (1993), 243–262.
- [5] P. Exner: *Leaky quantum graphs: a review*, Proceedings of the Isaac Newton Institute programme “Analysis on Graphs and Applications”, AMS “Proceedings of Symposia in Pure Mathematics” Series, vol. 77, Providence, R.I., 2008; pp. 523–564.
- [6] P. Exner, K. Yoshitomi: *Asymptotics of eigenvalues of the Schrödinger operator with a strong  $\delta$ -interaction on a loop*. J. Geom. Phys. **41** (2002), 344–358.
- [7] G. Grubb: *The mixed boundary value problem, Krein resolvent formulas and spectral asymptotic estimates*. J. Math. Anal. Appl. **382** (2011), 339–363.



- [8] M. Levitin, L. Parnowski: *On the principal eigenvalue of a Robin problem with a large parameter*. Math. Nachrichten **281** (2008), 272–281.
- [9] S. A. Nazarov: *An example of multiple gaps in the spectrum of a periodic waveguide*. Sbornik: Mathematics **201**:4 (2010), 569–594.
- [10] A. Posilicano: *Boundary triples and Weyl functions for singular perturbations of selfadjoint operator*, Meth. Funct. Anal. Topol. **10** (2004), 57–63.